

# Hermitian quasi-exactly solvable matrix Schrödinger operators

Stanislav Spichak\* and Renat Zhdanov†

Institute of Mathematics, 3 Tereshchenkivska Street, 252004 Kyiv, Ukraine

## Abstract

We construct six multi-parameter families of Hermitian quasi-exactly solvable matrix Schrödinger operators in one variable. The method for finding these operators relies heavily upon a special representation of the Lie algebra  $o(2, 2) \cong sl(2) \oplus sl(2)$  whose representation space contains an invariant finite-dimensional subspace. Besides that we give several examples of quasi-exactly solvable matrix models that have square-integrable eigenfunctions. These examples are in direct analogy with the quasi-exactly solvable scalar Schrödinger operators obtained by Turbiner and Ushveridze.

## I. Introduction

In the paper [1] we have suggested a generalization of the Turbiner-Shifman approach [2]–[4] to the construction of quasi-exactly solvable (QES) models on line for the case of matrix Hamiltonians. We remind that originally their method was applied to scalar one-dimensional stationary Schrödinger equations. Later on it was extended to the case of multi-dimensional scalar stationary Schrödinger equations [4]–[7] (see, also [8]).

A systematic description of our approach can be found in the paper [9]. The procedure of constructing a QES matrix (scalar) model is based on the concept of a Lie-algebraic Hamiltonian. We call a second-order operator in one variable Lie-algebraic if the following requirements are met:

- The Hamiltonian is a quadratic form with constant coefficients of first-order operators  $Q_1, Q_2, \dots, Q_n$  forming a Lie algebra  $g$ ;
- The Lie algebra  $g$  has a finite-dimensional invariant subspace  $\mathcal{I}$  of the whole representation space.

Now if a given Hamiltonian  $H[x]$  is Lie-algebraic, then after being restricted to the space  $\mathcal{I}$  it becomes a matrix operator  $\mathcal{H}$  whose eigenvalues and eigenvectors are computed in a purely algebraic way. This means that the Hamiltonian  $H[x]$  is quasi-exactly solvable (for further details on scalar QES models see [8]).

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\*Electronic mail: spichak@apmat.freenet.kiev.ua

†Electronic mail: rzhdanov@apmat.freenet.kiev.ua

It should be noted that there exist alternative approaches to constructing matrix QES models [10]–[15]. The principal idea of these is fixing the form of basis elements of the invariant space  $\mathcal{I}$ . They are chosen to be polynomials in  $x$ . This assumption leads to a challenging problem of classification of superalgebras by matrix-differential operators in one variable [15].

We impose no *a priori* restrictions on the form of basis elements of the space  $\mathcal{I}$ . What is fixed is the class to which the basis elements of the Lie algebra  $g$  should belong. Following [1, 9] we choose this class  $\mathcal{L}$  as the set of matrix differential operators of the form

$$\mathcal{L} = \{Q : Q = a(x)\partial_x + A(x)\}. \quad (1)$$

Here  $a(x)$  is a smooth real-valued function and  $A(x)$  is an  $N \times N$  matrix whose entries are smooth complex-valued functions of  $x$ . Hereafter we denote  $d/dx$  as  $\partial_x$ .

Evidently,  $\mathcal{L}$  can be treated as an infinite-dimensional Lie algebra with a standard commutator as a Lie bracket. Given a subalgebra  $\langle Q_1, Q_2, \dots, Q_n \rangle$  of the algebra  $\mathcal{L}$ , that has a finite-dimensional invariant space, we can easily construct a QES matrix model. To this end we compose a bilinear combination of the operators  $Q_1, Q_2, \dots, Q_n$  and of the unit  $N \times N$  matrix  $I$  with constant complex coefficients  $\alpha_{jk}$  and get

$$H[x]\psi(x) = \left( \sum_{j,k=1}^n \alpha_{jk} Q_j Q_k \right) \psi(x). \quad (2)$$

So there arises a natural problem of classification of subalgebras of the algebra  $\mathcal{L}$  within its inner automorphism group. The problem of classification of inequivalent realizations of Lie algebras on line and on plane has been solved in a full generality by Lie itself [16, 17] (see, also [18]). However, the classification problem for the case when  $A(x) \neq f(x)I$  with a scalar function  $f(x)$  is open by now. In the paper [9] we have classified realizations of the Lie algebras of the dimension up to three by the operators belonging to  $\mathcal{L}$  with an arbitrary  $N$ . Next, fixing  $N = 2$  we have studied which of them give rise to QES matrix Hamiltonians  $H[x]$ . It occurs that the only three-dimensional algebra that meets this requirement is the algebra  $sl(2)$  (which is fairly easy to predict taking into account the scalar case!). This yields the two families of  $2 \times 2$  QES models, one of them under proper restrictions giving rise to the well-known family of scalar QES Hamiltonians (for more details, see [9]).

As is well-known a physically meaningful QES matrix Schrödinger operator has to be Hermitian. This requirement imposes restrictions on the choice of QES models which somehow were beyond considerations of our previous papers [1, 9]. The principal aim of the present paper is to formulate and implement an efficient algebraic procedure for constructing QES Hermitian matrix Schrödinger operators

$$\hat{H}[x] = \partial_x^2 + V(x). \quad (3)$$

This requires a slight modification of the algebraic procedure used in [9]. We consider as an algebra  $g$  the direct sum of two  $sl(2)$  algebras which is equivalent to the algebra  $o(2, 2)$ . The necessary algebraic structures are introduced in Section 2. The next Section is devoted to constructing in a regular way Hermitian QES matrix Schrödinger operators on line. We give the list of thus obtained QES models in Section 4. The fifth Section contains a number of examples of Hermitian QES Schrödinger operators that have square integrable eigenfunctions.

## II. Extension of the algebra $sl(2)$

Following [1, 9] we consider the representation of the algebra  $sl(2)$

$$\begin{aligned} sl(2) &= \langle Q_-, Q_0, Q_+ \rangle \\ &= \langle \partial_x, x\partial_x - \frac{m-1}{2} + S_0, x^2\partial_x - (m-1)x + 2S_0x + S_+ \rangle, \end{aligned} \quad (4)$$

where  $S_0 = \sigma_3/2$ ,  $S_+ = (i\sigma_2 + \sigma_1)/2$ ,  $\sigma_k$  are the  $2 \times 2$  Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and  $m \geq 2$  is an arbitrary natural number. This representation gives rise to a family of QES models and furthermore the algebra (4) has the following finite-dimensional invariant space

$$\begin{aligned} \mathcal{I}_{sl(2)} &= \mathcal{I}_1 \oplus \mathcal{I}_2 = \langle \vec{e}_1, x\vec{e}_1, \dots, x^{m-2}\vec{e}_1 \rangle \oplus \\ &\langle m\vec{e}_2, \dots, mx^j\vec{e}_2 - jx^{j-1}\vec{e}_1, \dots, mx^m\vec{e}_2 - mx^{m-1}\vec{e}_1 \rangle. \end{aligned} \quad (5)$$

Since the spaces  $\mathcal{I}_1, \mathcal{I}_2$  are invariant with respect to an action of any of the operators (4), the above representation is reducible. A more serious trouble is that it is not possible to construct a QES operator, that is equivalent to a Hermitian Schrödinger operator, by taking a bilinear combination (2) of operators (4) with coefficients being complex numbers. To overcome this difficulty we use the idea indicated in [9] and let the coefficients of the bilinear combination (2) to be constant  $2 \times 2$  matrices. To this end we introduce a wider Lie algebra and add to the algebra (4) the following three matrix operators:

$$R_- = S_-, \quad R_0 = S_-x + S_0, \quad R_+ = S_-x^2 + 2S_0x + S_+, \quad (6)$$

where  $S_{\pm} = (i\sigma_2 \pm \sigma_1)/2$ .

It is straightforward to verify that the space (5) is invariant with respect to an action of a linear combination of the operators (6). Consider next the following set of operators:

$$\langle T_{\pm} = Q_{\pm} - R_{\pm}, T_0 = Q_0 - R_0, R_{\pm}, R_0, I \rangle, \quad (7)$$

where  $Q$  and  $R$  are operators (4) and (6), respectively, and  $I$  is a unit  $2 \times 2$  matrix. By a direct computation we check that the operators  $T_{\pm}, T_0$  as well as the operators  $R_{\pm}, R_0$ , fulfill the commutation relations of the algebra  $sl(2)$ . Furthermore any of the operators  $T_{\pm}, T_0$  commutes with any of the operators  $R_{\pm}, R_0$ . Consequently, operators (7) form the Lie algebra

$$sl(2) \oplus sl(2) \oplus I \cong o(2, 2) \oplus I.$$

In a sequel we denote this algebra as  $g$ .

The Casimir operators of the Lie algebra  $g$  are multiples of the unit matrix

$$C_1 = T_0^2 - T_+T_- - T_0 = \left( \frac{m^2 - 1}{4} \right) I, \quad K_2 = R_0^2 - R_+R_- - R_0 = \frac{3}{4} I.$$

Using this fact it can be shown that the representation of  $g$  realized on the space  $\mathcal{I}_{sl(2)}$  is irreducible.

One more remark is that the operators (7) satisfy the following relations:

$$\begin{aligned} R_-^2 &= 0, \quad R_0^2 = \frac{1}{4}, \quad R_+^2 = 0, \\ \{R_-, R_0\} &= 0, \quad \{R_+, R_0\} = 0, \quad \{R_-, R_+\} = -1, \\ R_- R_0 &= \frac{1}{2} R_-, \quad R_0 R_+ = \frac{1}{2} R_+, \quad R_- R_+ = R_0 - \frac{1}{2}. \end{aligned} \tag{8}$$

Here  $\{Q_1, Q_2\} = Q_1 Q_2 + Q_2 Q_1$ . One of the consequences of this fact is that the algebra  $g$  may be considered as a superalgebra which shows an evident link to the results of the paper [15].

### III. The general form of the Hermitian QES operator

Using the commutation relations of the Lie algebra  $g$  together with relations (8) one can show that any bilinear combination of the operators (7) is a linear combination of twenty one (basis) quadratic forms of the operators (7). Composing this linear combination yields all QES models which can be obtained with the help of our approach. However the final goal of the paper is not to get some families of QES matrix second-order operators as such but to get QES Schrödinger operators (3). This means that it is necessary to transform bilinear combination (2) to the standard form (3). What is more, it is essential that the corresponding transformation should be given by explicit formulae, since we need to write down explicitly the matrix potential  $V(x)$  of thus obtained QES Schrödinger operator and the basis functions of its invariant space.

The general form of QES model obtainable within the framework of our approach is as follows

$$H[x] = \xi(x) \partial_x^2 + B(x) \partial_x + C(x), \tag{9}$$

where  $\xi(x)$  is some real-valued function and  $B(x), C(x)$  are matrix functions of the dimension  $2 \times 2$ . Let  $U(x)$  be an invertible  $2 \times 2$  matrix-function satisfying the system of ordinary differential equations

$$U'(x) = \frac{1}{2\xi(x)} \left( \frac{\xi'(x)}{2} - B(x) \right) U(x), \tag{10}$$

and the function  $f(x)$  be defined by the relation

$$f(x) = \pm \int \frac{dx}{\sqrt{\xi(x)}}. \tag{11}$$

Then the change of variables reducing (9) to the standard form (3) reads as

$$\begin{aligned} x &\rightarrow y = f(x), \\ H[x] &\rightarrow \hat{H}[y] = \hat{U}^{-1}(y) H[f^{-1}(y)] \hat{U}(y), \end{aligned} \tag{12}$$

where  $f^{-1}$  stands for the inverse of  $f$  and  $\hat{U}(y) = U(f^{-1}(y))$ .

Performing the transformation (12) yields the Schrödinger operator

$$\hat{H}[y] = \partial_y^2 + V(y) \tag{13}$$

with

$$V(y) = \left\{ U^{-1}(x) \left[ -\frac{1}{4\xi} B^2(x) - \frac{1}{2} B'(x) + \frac{\xi'}{2\xi} B(x) + C(x) \right] U(x) + \frac{\xi''}{4} - \frac{3\xi'^2}{16\xi} \right\} \Big|_{x=f^{-1}(y)}. \quad (14)$$

Hereafter, the notation  $\{W(x)\}_{x=f^{-1}(y)}$  means that we should replace  $x$  with  $f^{-1}(y)$  in the expression  $W(x)$ .

Furthermore, if we denote the basis elements of the invariant space (5) as  $\vec{f}_1(x), \dots, \vec{f}_{2m}(x)$ , then the invariant space of the operator  $\hat{H}[y]$  takes the form

$$\hat{\mathcal{I}}_{sl(2)} = \langle \hat{U}^{-1}(y) \vec{f}_1(f^{-1}(y)), \dots, \hat{U}^{-1}(y) \vec{f}_{2m}(f^{-1}(y)) \rangle. \quad (15)$$

In view of the remark made at the beginning of this section we are looking for such QES models that the transformation law (12) can be given explicitly. This means that we should be able to construct a solution of system (10) in an explicit form. To achieve this goal we select from the above mentioned set of twenty one linearly independent quadratic forms of operators (7) the twelve forms,

$$\begin{aligned} A_0 &= \partial_x^2, \quad A_1 = x\partial_x^2, \quad A_2 = x^2\partial_x^2 + (m-1)\sigma_3, \\ B_0 &= \partial_x, \quad B_1 = x\partial_x + \frac{\sigma_3}{2}, \quad B_2 = x^2\partial_x - (m-1)x + \sigma_3x + \sigma_1, \\ C_1 &= \sigma_1\partial_x + \frac{m}{2}\sigma_3, \quad C_2 = i\sigma_2\partial_x + \frac{m}{2}\sigma_3, \quad C_3 = \sigma_3\partial_x, \\ D_1 &= x^3\partial_x^2 - 2\sigma_1x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ D_2 &= x^3\partial_x^2 - 2i\sigma_2x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ D_3 &= 2\sigma_3x\partial_x + (1-2m)\sigma_3, \end{aligned} \quad (16)$$

whose linear combinations have such a structure that system (10), can be integrated in a closed form. However, in the present paper we study systematically the first nine quadratic forms from the above list. The quadratic forms  $D_1, D_2, D_3$  are used to construct an example of QES model such that the matrix potential is expressed via the Weierstrass function.

Thus the general form of the Hamiltonian to be considered in a sequel is as follows

$$\begin{aligned} H[x] &= \sum_{\mu=0}^2 (\alpha_\mu A_\mu + \beta_\mu B_\mu) + \sum_{i=1}^3 \gamma_i C_i = (\alpha_2 x^2 + \alpha_1 x + \alpha_0) \partial_x^2 \\ &\quad + (\beta_2 x^2 + \beta_1 x + \beta_0 + \gamma_1 \sigma_1 + i\gamma_2 \sigma_2 + \gamma_3 \sigma_3) \partial_x + \beta_2 \sigma_3 x \\ &\quad - \beta_2 (m-1)x + \beta_2 \sigma_1 + \left[ \alpha_2 (m-1) + \frac{\beta_1}{2} + \frac{m}{2} (\gamma_1 + \gamma_2) \right] \sigma_3. \end{aligned} \quad (17)$$

Here  $\alpha_0, \alpha_1, \alpha_2$  are arbitrary real constants and  $\beta_0, \dots, \gamma_3$  are arbitrary complex constants.

If we denote

$$\begin{aligned} \tilde{\gamma}_1 &= \gamma_1, \quad \tilde{\gamma}_2 = i\gamma_2, \quad \tilde{\gamma}_3 = \gamma_3, \quad \delta = 2\alpha_2(m-1) + \beta_1 + m(\gamma_1 + \gamma_2), \\ \xi(x) &= \alpha_2 x^2 + \alpha_1 x + \alpha_0, \quad \eta(x) = \beta_2 x^2 + \beta_1 x + \beta_0, \end{aligned} \quad (18)$$

then the general solution of system (10) reads as

$$U(x) = \xi^{1/4}(x) \exp \left[ -\frac{1}{2} \int \frac{\eta(x)}{\xi(x)} dx \right] \exp \left[ -\frac{1}{2} \tilde{\gamma}_i \sigma_i \int \frac{1}{\xi(x)} dx \right] \Lambda, \quad (19)$$

where  $\Lambda$  is an arbitrary constant invertible  $2 \times 2$  matrix. Performing the transformation (12) with  $U(x)$  being given by (19) reduces QES operator (17) to a Schrödinger form (13), where

$$\begin{aligned} V(y) = & \left\{ \frac{1}{4\xi} \Lambda^{-1} \{ -\eta^2 + 2\xi'\eta - 2\xi\eta' - 4\beta_2(m-1)x\xi - \tilde{\gamma}_i^2 \right. \\ & + 2(\xi' - \eta)\tilde{\gamma}_i\sigma_i + 4\beta_2\xi U^{-1}(x)\sigma_1 U(x) + (4\beta_2x + 2\delta)\xi \\ & \left. \times U^{-1}(x)\sigma_3 U(x) \} \Lambda + \frac{\alpha_2}{2} - \frac{3(2\alpha_2x + \alpha_1)^2}{16\xi} \right\} \Big|_{x=f^{-1}(y)}. \end{aligned} \quad (20)$$

Here  $\xi, \eta$  are functions of  $x$  defined in (18) and  $f^{-1}(y)$  is the inverse of  $f(x)$  which is given by (11).

The requirement of hermiticity of the Schrödinger operator (13) is equivalent to the requirement of hermiticity of the matrix  $V(y)$ . To select from the multi-parameter family of matrices (20) Hermitian ones we will make use of the following technical lemmas.

**Lemma 1** *The matrices  $z\sigma_a, w(\sigma_a \pm i\sigma_b), a \neq b$ , with  $\{z, w\} \subset \mathbf{C}, z \notin \mathbf{R}, w \neq 0$  cannot be reduced to Hermitian matrices with the help of a transformation*

$$A \rightarrow A' = \Lambda^{-1} A \Lambda, \quad (21)$$

where  $\Lambda$  is an invertible constant  $2 \times 2$  matrix.

**Proof.** It is sufficient to prove the statement for the case  $a = 1, b = 2$ , since all other cases are equivalent to this one. Suppose the inverse, namely that there exists a transformation (21) transforming the matrix  $z\sigma_1$  to a Hermitian matrix  $A'$ . As  $\text{tr}(z\sigma_1) = \text{tr} A' = 0$ , the matrix  $A'$  has the form  $\alpha_i \sigma_i$  with some real constants  $\alpha_i$ . Next, from the equality  $\det(z\sigma_1) = \det A'$  we get  $z^2 = \alpha_i^2$ . The last relation is in contradiction to the fact that  $z \notin \mathbf{R}$ . Consequently, the matrix  $z\sigma_1$  cannot be reduced to a Hermitian matrix with the aid of a transformation (21).

Let us turn now to the matrix  $w(\sigma_1 + i\sigma_2)$ . Taking a general form of the matrix  $\Lambda$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we represent (21) as follows

$$A' = \Lambda^{-1} w(\sigma_1 + i\sigma_2) \Lambda = \frac{2w}{\delta} \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}, \quad \delta = \det \Lambda.$$

The conditions of hermiticity of the matrix  $A'$  read

$$\frac{w}{\delta} cd = \frac{\bar{w}}{\bar{\delta}} \bar{c} \bar{d}, \quad \frac{-w}{\delta} c^2 = \frac{\bar{w}}{\bar{\delta}} \bar{d}^2.$$

where the bar over a symbol stands for the complex conjugation.

It follows from the second relation that  $c, d$  can vanish only simultaneously which is impossible in view of the fact that the matrix  $\Lambda$  is invertible. Consequently, the relation  $cd \neq 0$  holds. Hence we get

$$\frac{-d}{c} = \frac{\bar{c}}{\bar{d}} \leftrightarrow |c|^2 + |d|^2 = 0.$$

This contradiction proves the fact that the matrix  $w(\sigma_1 + i\sigma_2)$  cannot be reduced to a Hermitian form.

As the matrix  $\sigma_1 + i\sigma_2$  is transformed to become  $\sigma_1 - i\sigma_2$  with the use of an appropriate transformation (21), the lemma is proved.

**Lemma 2** *Let  $\vec{a} = (a_1, a_2, a_3)$ ,  $\vec{b} = (b_1, b_2, b_3)$ ,  $\vec{c} = (c_1, c_2, c_3)$  be complex vectors and  $\vec{\sigma}$  be the vector whose components are the Pauli matrices  $(\sigma_1, \sigma_2, \sigma_3)$ . Then the following assertions holds true.*

1. *A non-zero matrix  $\vec{a}\vec{\sigma}$  is reduced to a Hermitian form with the help of a transformation (21) iff  $\vec{a}^2 > 0$  (this inequality means, in particular, that  $\vec{a}^2 \in \mathbf{R}$ );*
2. *Non-zero matrices  $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}$  with  $\vec{b} \neq \lambda\vec{a}$ ,  $\lambda \in \mathbf{R}$ , are reduced simultaneously to Hermitian forms with the help of a transformation (21) iff*

$$\vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0;$$

3. *Matrices  $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}, \vec{c}\vec{\sigma}$  with  $\vec{a} \neq \vec{0}$ ,  $\vec{b} \neq \lambda\vec{a}$ ,  $\vec{c} \neq \mu\vec{b}$ ,  $\{\lambda, \mu\} \subset \mathbf{R}$  are reduced simultaneously to Hermitian forms with the help of a transformation (21) iff*

$$\begin{aligned} \vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0, \\ \{\vec{a}\vec{c}, \quad \vec{b}\vec{c}, \quad (\vec{a} \times \vec{b})\vec{c}\} \subset \mathbf{R}. \end{aligned}$$

Here we designate the scalar product of vectors  $\vec{a}, \vec{b}$  as  $\vec{a}\vec{b}$  and the vector product of these as  $\vec{a} \times \vec{b}$ .

**Proof.** Let us first prove the necessity of the assertion 1 of the lemma. Suppose that the non-zero matrix  $\vec{a}\vec{\sigma}$  can be reduced to a Hermitian form. We will prove that hence it follows the inequality  $\vec{a}^2 > 0$ .

Consider the matrices:

$$\Lambda_{ij}(a, b) = \begin{cases} 1 + \epsilon_{ijk} \frac{\sqrt{a^2 + b^2} - b}{a} i\sigma_k, & a \neq 0, \\ 1, & a = 0, \end{cases} \quad (22)$$

where  $(i, j, k) = \text{cycle } (1, 2, 3)$ . It is not difficult to verify that these matrices are invertible, provided

$$\sqrt{a_i^2 + a_j^2} \neq 0. \quad (23)$$

Given the condition (23), the following relations hold

$$\sigma_l \rightarrow \Lambda_{ij}^{-1}(a, b) \sigma_l \Lambda_{ij}(a, b) = \begin{cases} \sigma_k, & l = k, \\ \frac{b\sigma_i + a\sigma_j}{\sqrt{a^2 + b^2}}, & l = i, \\ \frac{-a\sigma_i + b\sigma_j}{\sqrt{a^2 + b^2}}, & l = j. \end{cases} \quad (24)$$

As  $\vec{a}$  is a non-zero vector, there exists at least one pair of the indices  $i, j$  such that  $a_i^2 + a_j^2 \neq 0$ . Applying the transformation (24) with  $a = a_i, b = a_j$  we get

$$\vec{a}\vec{\sigma} \rightarrow \vec{a}'\vec{\sigma} = \sqrt{a_i^2 + a_j^2} \sigma_j + a_k \sigma_k \quad (25)$$

(no summation over the indices  $i, j, k$  is carried out). As the direct check shows, the quantity  $\vec{a}^2$  is invariant with respect to transformation (24), i.e.  $\vec{a}^2 = \vec{a}'^2$ .

If  $\vec{a}^2 = 0$ , then  $a_j'^2 + a_k'^2 = 0$ , or  $a_i' = \pm i a_k'$ . Hence by force of Lemma 1 it follows that the matrix (25) cannot be reduced to a Hermitian form. Consequently,  $\vec{a}^2 \neq 0$  and the relation  $a_j'^2 + a_k'^2 \neq 0$  holds true. Applying transformation (24) with  $a = \sqrt{a_i^2 + a_j^2}, b = a_k$  we get

$$\vec{a}'\vec{\sigma} \rightarrow \sqrt{\vec{a}^2} \sigma_k. \quad (26)$$

Due to Lemma 1, if the number  $\sqrt{\vec{a}^2}$  is complex, then the above matrix cannot be transformed to a Hermitian matrix. Consequently, the relation  $\vec{a}^2 > 0$  holds true.

The sufficiency of the assertion 1 of the lemma follows from the fact that, given the condition  $\vec{a}^2 > 0$ , the matrix (26) is Hermitian.

Now we will prove the necessity of the assertion 2 of the lemma. First of all we note that due to assertion 1,  $\vec{a}^2 > 0, \vec{b}^2 > 0$ . Next, without loss of generality we can again suppose that  $a_i^2 + a_j^2 \neq 0$ . Taking the superposition of two transformations of the form (24) with  $a = a_i, b = a_j$  and  $a = \sqrt{a_i^2 + a_j^2}, b = a_k$  yields

$$\begin{aligned} \Lambda_{ij}(a_i, a_j) \Lambda_{jk}(\sqrt{a_i^2 + a_j^2}, a_k) &= 1 + i\epsilon_{ijk} \frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2 + a_j^2}} \sigma_i \\ &+ i\epsilon_{ijk} \frac{\sqrt{a_i^2 + a_j^2} - a_j}{a_i} \sigma_k - i\epsilon_{ijk} \frac{\sqrt{a_i^2 + a_j^2} - a_j}{a_i} \frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2 + a_j^2}} \sigma_j \end{aligned} \quad (27)$$

(here the finite limit exists when  $a_i \rightarrow 0$ ). Using this formula and taking into account (24) yield

$$\vec{a}\vec{\sigma} \rightarrow \sqrt{\vec{a}^2} \sigma_k, \quad \vec{b}\vec{\sigma} \rightarrow \vec{b}'\vec{\sigma} = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}} \sigma_i + \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2} \sqrt{a_i^2 + a_j^2}} \sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}} \sigma_k. \quad (28)$$

Let us show that the necessary condition for the matrices  $\sqrt{\vec{a}^2} \sigma_k, \vec{b}'\vec{\sigma}$  to be reducible to Hermitian forms simultaneously reads as  $\vec{a}\vec{b} \in \mathbf{R}$ . Indeed, as the matrices  $\vec{b}'\vec{\sigma}, \sigma_k$  are simultaneously reduced to Hermitian forms, the matrix  $\vec{b}'\vec{\sigma} + \lambda \sigma_k$  can be reduced to a Hermitian form with any real  $\lambda$ . Hence, in view of the assertion 1 we conclude that

$$b_i'^2 + b_j'^2 + (b_k' + \lambda)^2 > 0, \quad (29)$$



where  $\lambda$  is an arbitrary real number. The above equality may be valid only when  $b'_k = \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}} \in \mathbf{R}$ .

Choosing  $\lambda = -b'_k$  in (29) yields that  $b_i'^2 + b_j'^2 > 0$ . Since  $b_i'^2 + b_j'^2 = (\vec{a} \times \vec{b})^2$ , hence we get the desired inequality  $(\vec{a} \times \vec{b})^2 > 0$ . The necessity is proved.

In order to prove the sufficiency of the assertion 2, we consider transformation (24) with

$$a = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}}, \quad b = \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2} \sqrt{a_i^2 + a_j^2}}. \quad (30)$$

This transformation leaves the matrix  $\sqrt{\vec{a}^2} \sigma_k$  invariant, while the matrix  $\vec{b}' \vec{\sigma}$  (28) transforms as follows

$$\vec{b}' \vec{\sigma} \rightarrow \vec{b}'' \vec{\sigma} = \frac{\sqrt{(\vec{a} \times \vec{b})^2}}{\sqrt{\vec{a}^2}} \sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}} \sigma_k, \quad (31)$$

whence it follows the sufficiency of the assertion 2.

The proof of the assertion 3 of the lemma is similar to one of the assertion 2. The first three conditions are obtained with account of the assertion 2. A sequence of transformations (24) with  $a, b$  of the form (27), (30) transforms the matrix  $\vec{c} \vec{\sigma}$  to become

$$\vec{c} \vec{\sigma} \rightarrow \vec{c}'' \vec{\sigma} = \frac{\epsilon_{ijk} \vec{a}(\vec{c} \times \vec{b})}{\sqrt{(\vec{c} \times \vec{b})^2}} \sigma_i + \frac{(\vec{a} \times \vec{b})(\vec{a} \times \vec{c})}{\sqrt{(\vec{c} \times \vec{b})^2} \sqrt{\vec{a}^2}} \sigma_j + \frac{\vec{a}\vec{c}}{\sqrt{\vec{a}^2}} \sigma_k. \quad (32)$$

Using the standard identities for the mixed vector products we establish that the coefficients by the matrices  $\sigma_i, \sigma_j, \sigma_k$  are real if and only if the relations

$$\{\vec{a}\vec{c}, \vec{b}\vec{c}, (\vec{a} \times \vec{b})\vec{c}\} \subset \mathbf{R}$$

hold true. This completes the proof of Lemma 2.

Lemma 2 plays the crucial role when reducing the potentials (20), to Hermitian forms. This is done as follows. Firstly, we reduce the QES operator to the Schrödinger form

$$\partial_y^2 + f(y) \vec{a} \vec{\sigma} + g(y) \vec{b} \vec{\sigma} + h(y) \vec{c} \vec{\sigma} + r(y).$$

Here  $f, g, h, r$  are some linearly-independent scalar functions and  $\vec{a} = (a_1, a_2, a_3)$ ,  $\vec{b} = (b_1, b_2, b_3)$ ,  $\vec{c} = (c_1, c_2, c_3)$  are complex constant vectors whose components depend on the parameters  $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ . Next, using Lemma 2 we obtain the conditions for the parameters  $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$  that provide a simultaneous reducibility of the matrices  $\vec{a} \vec{\sigma}, \vec{b} \vec{\sigma}, \vec{c} \vec{\sigma}$  to Hermitian forms. Then, making use of formulae (22), (27), (30) we find the form of the matrix  $\Lambda$ . Formulae (26), (31), (32) yield explicit forms of the transformed matrices  $\vec{a} \vec{\sigma}, \vec{b} \vec{\sigma}, \vec{c} \vec{\sigma}$  and, consequently, the Hermitian form of the matrix potential  $V(y)$ .

## IV. QES matrix models

Applying the algorithm mentioned at the end of the previous section we have obtained a complete description of QES matrix models (17) that can be reduced to Hermitian Schrödinger matrix operators. We list below the final result, namely, the restrictions on the choice of parameters

and the explicit forms of the QES Hermitian Schrödinger operators and then consider in some detail a derivation of the corresponding formulae for one of the six inequivalent cases. In the formulae below we denote the disjunction of two statements  $A$  and  $B$  as  $[A] \vee [B]$ .

**Case 1.**  $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}_3 = 0$  and

$$[\beta_0, \beta_1, \beta_2 \in \mathbf{R}] \vee [\beta_2 = 0, \beta_1 = 2\alpha_2, \beta_0 = \alpha_1 + i\mu, \mu \in \mathbf{R}];$$

$$\begin{aligned} \hat{H}[y] = & \partial_y^2 + \left\{ \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \{ -\beta_2^2 x^4 - [2\beta_1 \beta_2 + 4\alpha_2 \beta_2 (m-1)] x^3 \right. \\ & + [2\alpha_2 \beta_1 - 2\alpha_1 \beta_2 - \beta_1^2 - 2\beta_0 \beta_2 - 4\alpha_1 \beta_2 (m-1)] x^2 \\ & + [4\alpha_2 \beta_0 - 2\beta_0 \beta_1 - 4m\alpha_0 \beta_2] x + 2\alpha_1 \beta_0 - 2\alpha_0 \beta_1 - \beta_0^2 \\ & + 4\beta_2 (\alpha_2 x^2 + \alpha_1 x + \alpha_0) \sigma_1 + (4\beta_2 x + 2\delta) (\alpha_2 x^2 + \alpha_1 x + \alpha_0) \sigma_3 \} \\ & \left. + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \right\} \Big|_{x=f^{-1}(y)}, \\ \Lambda = & 1. \end{aligned}$$

**Case 2.**  $\beta_2, \delta = 0$  and

$$\begin{aligned} & 2\alpha_2 \beta_1 - \beta_1^2 \in \mathbf{R}, \quad 2\alpha_2 \beta_0 - \beta_0 \beta_1 \in \mathbf{R}, \quad 2\alpha_1 \beta_0 - 2\beta_1 \alpha_0 - \beta_0^2 - \tilde{\gamma}_i^2 \in \mathbf{R}, \\ & [(2\alpha_2 - \beta_1)^2 \tilde{\gamma}_i^2 > 0] \vee [2\alpha_2 - \beta_1 = 0], \quad [(\alpha_1 - \beta_0)^2 \tilde{\gamma}_i^2 > 0] \vee [\alpha_1 - \beta_0 = 0]; \end{aligned}$$

$$\begin{aligned} \hat{H}[y] = & \partial_y^2 + \left\{ \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \left\{ \beta_1 (2\alpha_2 - \beta_1) x^2 + 2\beta_0 (2\alpha_2 - \beta_1) x \right. \right. \\ & + 2\alpha_1 \beta_0 - 2\beta_1 \alpha_0 - \beta_0^2 - \tilde{\gamma}_i^2 + [2(2\alpha_2 - \beta_1) x + 2(\alpha_1 - \beta_0)] \sqrt{\tilde{\gamma}_i^2} \sigma_3 \} \\ & \left. + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \right\} \Big|_{x=f^{-1}(y)}, \\ \Lambda = & \Lambda_{12}(\tilde{\gamma}_1, \tilde{\gamma}_2) \Lambda_{23}(\sqrt{\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2}, \tilde{\gamma}_3), \quad \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 \neq 0. \end{aligned}$$

(If  $\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 = 0$ , then one can choose another matrix  $\Lambda$  (27) with  $\tilde{\gamma}_i^2 + \tilde{\gamma}_j^2 \neq 0$ ).

**Case 3.**  $\alpha_2 \neq 0, \beta_2 \neq 0$  and

$$\begin{aligned} & [\{\beta_2, \gamma_1\} \subset \mathbf{Re}, \quad \gamma_3 = 0, \quad \gamma_2 = \sqrt{\gamma_1^2 - 2\alpha_2 \gamma_1}, \quad \alpha_2 \gamma_1 < 0, \\ & \beta_1 = 2\alpha_2 + \beta_2 \frac{\alpha_1}{\alpha_2}, \quad \beta_0 = \alpha_1 + \beta_2 \frac{\alpha_0}{\alpha_2}]; \end{aligned}$$

$$\begin{aligned} \hat{H}[y] = & \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \left\{ -\beta_2^2 x^4 \right. \right. \\ & \left. \left. - \left[ 2\beta_2^2 \frac{\alpha_1}{\alpha_2} + 4\alpha_2 \beta_2 m \right] x^3 - \left[ \frac{\beta_2^2}{\alpha_2^2} (\alpha_1^2 + 2\alpha_0 \alpha_2) + 2\alpha_1 \beta_2 (1 + 2m) \right] x^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \left[ \frac{2\alpha_1\beta_2(\alpha_1\alpha_2 + \alpha_0\beta_2)}{\alpha_2^2} + 4\alpha_0\beta_2m \right] x + \alpha_1^2 - \beta_2^2 \frac{\alpha_0^2}{\alpha_2^2} - 4\beta_2 \frac{\alpha_0\alpha_1}{\alpha_2} - 4\alpha_0\alpha_2 - 2\alpha_2\gamma_1 \\
& + 4\beta_2x(\alpha_2x^2 + \alpha_1x + \alpha_0) \left[ \sin \left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \sigma_1 + \cos \left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \sigma_3 \right] \\
& + 2(\alpha_2x^2 + \alpha_1x + \alpha_0) \left[ \frac{\sin \left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right)}{\sqrt{-2\alpha_2\gamma_1}} \left( \delta\sqrt{-2\alpha_2\gamma_1}\sigma_1 - 2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}\sigma_3 \right) \right. \\
& \left. + \cos \left( \theta(y)\sqrt{-2\alpha_2\gamma_1} \right) \left( \frac{2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}}{\sqrt{-2\alpha_2\gamma_1}}\sigma_1 + \delta\sigma_3 \right) \right] \Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1 + \left( \sqrt{1 - \frac{2\alpha_2}{\gamma_1}} - \sqrt{\frac{-2\alpha_2}{\gamma_1}} \right) \sigma_3.
\end{aligned}$$

**Case 4.**  $\alpha_2 \neq 0, \beta_2 = 0$ .

**Subcase 4.1.**  $\delta \neq 0, \gamma_1, \gamma_2$  do not vanish simultaneously and

$$\gamma_1^2 - \gamma_2^2 < 0, \quad \gamma_3 = i\mu, \quad \{\mu, \delta\} \subset \mathbf{R}, \quad i(\alpha_1 - \beta_0) \in \mathbf{R}, \quad \beta_1 = 2\alpha_2;$$

$$\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2x + \alpha_1)^2}{16(\alpha_2x^2 + \alpha_1x + \alpha_0)} + \frac{1}{4\xi} \left\{ -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \tilde{\gamma}_i^2 \right. \right. \\
&+ 2(\alpha_2x^2 + \alpha_1x + \alpha_0) \left[ \delta\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_1 \frac{\sin \left( \theta(y)\sqrt{-\tilde{\gamma}_i^2} \right)}{\sqrt{-\tilde{\gamma}_i^2}} \right. \\
&+ \left. \left. \frac{-i\delta\gamma_3\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_2 + \delta(\gamma_1^2 - \gamma_2^2)\sigma_3}{\tilde{\gamma}_i^2} \cos \left( \theta(y)\sqrt{-\tilde{\gamma}_i^2} \right) \right] + \left[ \frac{2\delta\alpha_2\gamma_3}{\tilde{\gamma}_i^2}x^2 \right. \right. \\
&+ \left. \left. \frac{2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2}x + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2} \right] \left( i\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_2 + \gamma_3\sigma_3 \right) \right\} \Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= \Lambda_{21}(i\gamma_1, \gamma_2).
\end{aligned}$$

**Subcase 4.2.**  $\delta \neq 0, \gamma_1 = \gamma_2 = 0, \gamma_3 \neq 0$  and

$$\{\delta, \beta_1(2\alpha_2 - \beta_1), \beta_0(2\alpha_2 - \beta_1), -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1, \gamma_3(2\alpha_2 - \beta_1), \gamma_3(\alpha_1 - \beta_0)\} \subset \mathbf{R};$$

$$\begin{aligned}
\hat{H}[y] &= \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2x + \alpha_1)^2}{16(\alpha_2x^2 + \alpha_1x + \alpha_0)} + \frac{1}{4(\alpha_2x^2 + \alpha_1x + \alpha_0)} \{ \beta_1(2\alpha_2 - \beta_1)x^2 \right. \\
&+ 2\beta_0(2\alpha_2 - \beta_1)x - \beta_0^2 + 2\alpha_1\beta_0 - 2\beta_1\alpha_0 - \gamma_3^2 \\
&+ \left. \left[ 2\delta\alpha_2x^2 + 2x((2\alpha_2 - \beta_1)\gamma_3 + \delta\alpha_1) + 2(\alpha_1 - \beta_0)\gamma_3 + 2\delta\alpha_0 \right] \sigma_3 \right\} \Bigg|_{x=f^{-1}(y)}, \\
\Lambda &= 1,
\end{aligned}$$

**Case 5.**  $\alpha_2 = 0, \beta_2 \neq 0$  and

$$\begin{aligned}
& \alpha_1 \neq 0, \quad \gamma_1^2 - \gamma_2^2 < 0, \quad \tilde{\gamma}_i^2 < 0, \quad \gamma_3 = \frac{\tilde{\gamma}_i^2}{2\alpha_1}, \\
& \{\beta_0, \beta_1, \beta_2, \gamma_2, \delta(\gamma_2^2 - \gamma_1^2) + 2\beta_2\gamma_1\gamma_3\} \subset \mathbf{R}, \\
& \{i(2\alpha_0\beta_2\gamma_3 - \beta_1\tilde{\gamma}_i^2 + 2\beta_2\alpha_1\gamma_1 + \delta\alpha_1\gamma_3), i((\alpha_1 - \beta_0)\tilde{\gamma}_i^2 + 2\beta_2\alpha_0\gamma_1 + \delta\alpha_0\gamma_3)\} \subset \mathbf{R};
\end{aligned}$$

$$\begin{aligned}
\hat{H}[y] = & \partial_y^2 + \left\{ -\frac{3\alpha_1^2}{16(\alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_1 x + \alpha_0)} \left\{ -\beta_2^2 x^4 - 2\beta_1 \beta_2 x^3 \right. \right. \\
& + \left[ (2 - 4m)\alpha_1 \beta_2 - \beta_1^2 - 2\beta_0 \beta_2 \right] x^2 - [2\beta_0 \beta_1 + 4m\alpha_0 \beta_2] x \\
& + 2\alpha_1 \beta_0 - 2\alpha_0 \beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 + 4x(\alpha_1 x + \alpha_0) \left[ \beta_2 \sqrt{\gamma_2^2 - \gamma_1^2} \sigma_1 \frac{\sin \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right)}{\sqrt{-\tilde{\gamma}_i^2}} \right. \\
& + \left. \left. \frac{\beta_2 \sqrt{(\gamma_1^2 - \gamma_2^2) \tilde{\gamma}_i^2}}{\tilde{\gamma}_i^2} \sigma_3 \cos \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right) \right] + 2(\alpha_1 x + \alpha_0) \right. \\
& \times \left[ \left( \frac{\delta(\gamma_2^2 - \gamma_1^2) + 2\beta_2 \gamma_1 \gamma_3}{\sqrt{\gamma_2^2 - \gamma_1^2}} \sigma_1 - \frac{2\beta_2 \gamma_2 \tilde{\gamma}_i^2}{\sqrt{(\gamma_1^2 - \gamma_2^2) \tilde{\gamma}_i^2}} \sigma_3 \right) \frac{\sin \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right)}{\sqrt{-\tilde{\gamma}_i^2}} \right. \\
& + \left. \left( \frac{2\beta_2 \gamma_2}{\sqrt{\gamma_2^2 - \gamma_1^2}} \sigma_1 + \frac{\delta(\gamma_1^2 - \gamma_2^2) - 2\beta_2 \gamma_1 \gamma_3}{\sqrt{(\gamma_1^2 - \gamma_2^2) \tilde{\gamma}_i^2}} \sigma_3 \right) \cos \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right) \right] \\
& + \left[ x \frac{4\alpha_0 \beta_2 \gamma_3 - 2\beta_1 \tilde{\gamma}_i^2 + 4\alpha_1 \beta_2 \gamma_1 + 2\delta \alpha_1 \gamma_3}{\tilde{\gamma}_i^2} \right. \\
& + \left. \left. \frac{(2\alpha_1 - 2\beta_0) \tilde{\gamma}_i^2 + 4\alpha_0 \beta_2 \gamma_1 + 2\delta \alpha_0 \gamma_3}{\tilde{\gamma}_i^2} \right] \left( -i \sqrt{-\tilde{\gamma}_i^2} \sigma_2 \right) \right\} \Big|_{x=f^{-1}(y)}, \\
\Lambda = & \Lambda_{21}(i\gamma_1, \gamma_2) \Lambda_{23} \left( -i\gamma_3 \sqrt{\gamma_2^2 - \gamma_1^2}, \gamma_1^2 - \gamma_2^2 \right).
\end{aligned}$$

**Case 6.**  $\alpha_2 = 0$ ,  $\beta_2 = 0$ .

**Subcase 6.1.**  $\delta \neq 0$ ,  $\tilde{\gamma}_1, \tilde{\gamma}_2$  do not vanish simultaneously and

$$\begin{aligned}
& \tilde{\gamma}_i^2 < 0, \{ \delta^2(\gamma_1^2 - \gamma_2^2) < 0, \beta_0, \beta_1 \} \subset \mathbf{R}, \\
& \{ i(-\beta_1 \tilde{\gamma}_i^2 + \delta \alpha_1 \gamma_3), i((\alpha_1 - \beta_0) \tilde{\gamma}_i^2 + \delta \alpha_0 \gamma_3) \} \subset \mathbf{R};
\end{aligned}$$

$$\begin{aligned}
\hat{H}[y] = & \partial_y^2 + \left\{ -\frac{3\alpha_1^2}{16(\alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_1 x + \alpha_0)} \left\{ -\beta_1^2 x^2 - 2\beta_0 \beta_1 x + 2\alpha_1 \beta_0 \right. \right. \\
& - 2\alpha_0 \beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 + 2(\alpha_1 x + \alpha_0) \left[ \delta \sqrt{\gamma_2^2 - \gamma_1^2} \sigma_1 \frac{\sin \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right)}{\sqrt{-\tilde{\gamma}_i^2}} \right. \\
& + \frac{\delta(\gamma_1^2 - \gamma_2^2)}{\sqrt{(\gamma_1^2 - \gamma_2^2) \tilde{\gamma}_i^2}} \sigma_3 \cos \left( \theta(y) \sqrt{-\tilde{\gamma}_i^2} \right) \Big] + \left[ x \frac{-2\beta_1 \tilde{\gamma}_i^2 + 2\delta \alpha_1 \gamma_3}{\tilde{\gamma}_i^2} \right. \\
& + \left. \left. \frac{(2\alpha_1 - 2\beta_0) \tilde{\gamma}_i^2 + 2\delta \alpha_0 \gamma_3}{\tilde{\gamma}_i^2} \right] \left( -i \sqrt{-\tilde{\gamma}_i^2} \sigma_2 \right) \right\} \Big|_{x=f^{-1}(y)}, \\
\Lambda = & \Lambda_{21}(i\gamma_1, \gamma_2) \Lambda_{23} \left( -i\gamma_3 \sqrt{\gamma_2^2 - \gamma_1^2}, \gamma_1^2 - \gamma_2^2 \right).
\end{aligned}$$

**Subcase 6.2.**

$$\begin{aligned} \gamma_1 = \gamma_2 = 0, \quad \gamma_3 \neq 0, \quad \{\beta_1^2, \beta_0\beta_1\} \subset \mathbf{R}, \\ \{-\beta_1\gamma_3 + \delta\alpha_1, (\alpha_1 - \beta_0)\gamma_3 + \delta\alpha_0, -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1\} \subset \mathbf{R}; \end{aligned}$$

$$\begin{aligned} \hat{H}[y] = & \partial_y^2 + \left\{ -\frac{3\alpha_1^2}{16(\alpha_1x + \alpha_0)} \right. \\ & + \frac{1}{4(\alpha_1x + \alpha_0)} \{ -\beta_1^2x^2 - 2\beta_0\beta_1x + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 - \gamma_3^2 \\ & \left. + 2(\alpha_1x + \alpha_0)[2x\beta_1(\alpha_1 - \gamma_3) + 2(\alpha_1 - \beta_0)\gamma_3 + 2\beta_1\alpha_0]\sigma_3 \} \right\} \Big|_{x=f^{-1}(y)}, \end{aligned}$$

$$\Lambda = 1.$$

In the above formulae we denote the inverse of the function

$$y = f(x) \equiv \int \frac{dx}{\sqrt{\alpha_2x^2 + \alpha_1x + \alpha_0}}, \quad (33)$$

as  $f^{-1}(y)$  and, what is more, the function  $\theta = \theta(y)$  is defined as follows

$$\theta(y) = - \left\{ \int \frac{dx}{\alpha_2x^2 + \alpha_1x + \alpha_0} \right\} \Big|_{x=f^{-1}(y)}, \quad (34)$$

and  $\tilde{\gamma}_i^2$  stands for  $\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2$ .

The whole procedure of derivation of the above formulae is very cumbersome. That is why we restrict ourselves to indicating the principal steps of the derivation of the corresponding formulae for the case when  $\alpha_2 \neq 0, \beta_2 \neq 0$  omitting the secondary details. It is not difficult to prove that  $\tilde{\gamma}_i^2 \neq 0$ . Indeed, suppose that the relation  $\tilde{\gamma}_i^2 = 0$  holds and consider the expression  $\Omega = U^{-1}(x)\sigma_3U(x)$  from (20). Making use of the Campbell-Hausdorff formula we get

$$\Omega = \sigma_3 + \theta(i\gamma_1\sigma_2 + \gamma_2\sigma_1) - \frac{\theta^2}{2}\gamma_3\tilde{\gamma}_i\sigma_i,$$

where  $\theta$  is the function (34). Considering the coefficient at  $\theta^2$ , yields that  $\gamma_3 = 0$  (otherwise using Lemma 2 we get the inequality  $\gamma_3^2\tilde{\gamma}_i^2 \neq 0$  that contradicts to the assumption  $\tilde{\gamma}_i^2 = 0$ ). Since the matrix coefficient at  $\theta$  has to be Hermitian, we get  $\tilde{\gamma}_i^2 = \gamma_1^2 - \gamma_2^2 < 0$ . This contradiction proves that  $\tilde{\gamma}_i^2 \neq 0$ . Taking into account the proved fact we represent the matrix potential (20) as follows

$$\begin{aligned} V(y) = & \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2x + \alpha_1)^2}{16(\alpha_2x^2 + \alpha_1x + \alpha_0)} \right. \\ & + \frac{1}{4(\alpha_2x^2 + \alpha_1x + \alpha_0)} \Lambda^{-1} \left\{ -\beta_2^2x^4 - [2\beta_1\beta_2 + 4\alpha_2\beta_2(m-1)]x^3 \right. \\ & \left. \left. + [2\alpha_2\beta_1 - 2\alpha_1\beta_2 - \beta_1^2 - 2\beta_0\beta_2 - 4\alpha_1\beta_2(m-1)]x^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& + [4\alpha_2\beta_0 - 2\beta_0\beta_1 - 4m\alpha_0\beta_2]x + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 \\
& + 4x(\alpha_2x^2 + \alpha_1x + \alpha_0) \left[ \beta_2\gamma_3(\tilde{\gamma}_i^2)^{-1}\tilde{\gamma}_i\sigma_i + \beta_2(\gamma_2\sigma_1 + i\gamma_1\sigma_2)(\tilde{\gamma}_i^2)^{-1/2} \sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \right. \\
& \left. + [\beta_2(-\gamma_1\gamma_3\sigma_1 - i\gamma_2\gamma_3\sigma_2 + (\gamma_1^2 - \gamma_2^2)\sigma_3)](\tilde{\gamma}_i^2)^{-1} \cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \right] \quad (35) \\
& + 2(\alpha_2x^2 + \alpha_1x + \alpha_0) \left[ (\delta\gamma_2\sigma_1 + i(\delta\gamma_1 - 2\beta_2\gamma_3)\sigma_2 - 2\beta_2\gamma_2\sigma_3)(\tilde{\gamma}_i^2)^{-1/2} \sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \right. \\
& + [(2\beta_2(\gamma_3^2 - \gamma_2^2) - \delta\gamma_1\gamma_3)\sigma_1 - i(2\beta_2\gamma_1\gamma_2 + \delta\gamma_2\gamma_3)\sigma_2 + (\delta(\gamma_1^2 - \gamma_2^2) - 2\beta_2\gamma_1\gamma_3)\sigma_3] \\
& \left. (\tilde{\gamma}_i^2)^{-1} \cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) \right] + [(-2\beta_2\tilde{\gamma}_i^2 + 4\alpha_2\beta_2\gamma_1 + 2\delta\alpha_2\gamma_3)x^2 + ((4\alpha_2 - 2\beta_1)\tilde{\gamma}_i^2 \\
& + 4\alpha_1\beta_2\gamma_1 + 2\delta\alpha_1\gamma_3)x + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 4\alpha_0\beta_2\gamma_1 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}](\tilde{\gamma}_i^2)^{-1}\tilde{\gamma}_i\sigma_i \Big\} \Lambda \Bigg|_{x=f^{-1}(y)},
\end{aligned}$$

where  $\theta = \theta(y)$  is given by (34).

Let us first suppose that  $\gamma_1, \gamma_2$  do not vanish simultaneously. We will prove that hence it follows that  $\tilde{\gamma}_i^2 \in \mathbf{R}$ . Consider the (non-zero) matrix coefficient at  $4x\xi \cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$  in the expression (35) and suppose that  $\sqrt{\tilde{\gamma}_i^2} = a + ib$ , with some non-zero real numbers  $a$  and  $b$ . Now it is easy to prove that  $\cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right) = f(x) + ig(x)$ , where  $f, g$  are linearly-independent real-valued functions. Considering the matrix coefficients of  $f(x), g(x)$  we see that in order to reduce the matrix (35) to a Hermitian form we should reduce to Hermitian forms the matrices  $A, iA$  which is impossible. This contradiction proves that  $\tilde{\gamma}_i^2 \in \mathbf{R}$ .

Consider next the non-zero matrix coefficients of  $4x\xi \frac{\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)}{\sqrt{\tilde{\gamma}_i^2}}, 4x\xi \cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$  in (35).

These coefficients can be represented in the form  $\vec{a}\vec{\sigma}, \vec{b}\vec{\sigma}$ , where

$$\vec{a} = \beta_2(\gamma_2, i\gamma_1, 0), \quad \vec{b} = \beta_2(-\gamma_1\gamma_3, -i\gamma_2\gamma_3, \gamma_1^2 - \gamma_2^2),$$

and, what is more,

$$\vec{a} \times \vec{b} = \beta_2^2(\gamma_1^2 - \gamma_2^2)(i\gamma_1, -\gamma_2, i\gamma_3).$$

Applying Lemma 2 yields

$$\beta_i \in \mathbf{R}, \quad \gamma_3 = i\mu, \quad \mu \in \mathbf{R}, \quad \gamma_1^2 - \gamma_2^2 < 0.$$

Next we turn to the matrix coefficient of  $2\xi \frac{\sinh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)}{\sqrt{\tilde{\gamma}_i^2}}$  which is of the form  $\vec{c}\vec{\sigma}$  with  $\vec{c} = (\delta\gamma_2, i(\delta\gamma_1 - 2\beta_2\gamma_3), -2\beta_2\gamma_2)$ . Making use of the assertion 3 of Lemma 2 we obtain the conditions

$$\{\gamma_1, \gamma_2\} \subset \mathbf{R}, \quad [\gamma_1 = 0] \vee [\gamma_3 = 0].$$

Considering in a similar way the matrix coefficient of  $2\xi \cosh\left(\theta\sqrt{\tilde{\gamma}_i^2}\right)$  yields the following restrictions on the coefficients  $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ :

$$\left[ \{\beta_2, \gamma_1\} \subset \mathbf{R}, \gamma_3 = 0, \gamma_2 = \sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}, \alpha_2\gamma_1 < 0, \beta_1 = 2\alpha_2 + \beta_2\frac{\alpha_1}{\alpha_2}, \beta_0 = \alpha_1 + \beta_2\frac{\alpha_0}{\alpha_2} \right].$$

As a result we get the formulae of Case 2.

One can prove in an analogous way that, provided  $\gamma_1 = \gamma_2 = 0, \gamma_3 \neq 0$ , the matrix (35) cannot be reduced to a Hermitian form.

## V. Some examples.

In this section we give several examples of Hermitian QES matrix Schrödinger operators that have a comparatively simple form and, furthermore, are in direct analogy to QES scalar Schrödinger operators.

*Example 1.* Let us consider Case I of the previous section with  $\alpha_0 = \beta_2 = 1$ , the remaining coefficients being equal to zero. This choice of parameters yields the following Hermitian QES matrix Schrödinger operator:

$$\hat{H}[y] = \partial_y^2 - \frac{1}{4}y^4 - my + \sigma_3 y + \sigma_1. \quad (36)$$

The invariant space  $\mathcal{I}$  of the above Schrödinger operator has the dimension  $2m$  and is spanned by the vectors

$$\vec{f}_j = \exp\left(\frac{y^3}{6}\right) \vec{e}_1 y^j, \quad \vec{g}_k = \exp\left(\frac{y^3}{6}\right) (m\vec{e}_2 y^k - k\vec{e}_y x^{k-1}),$$

where  $j = 0, \dots, m-2, k = 0, \dots, m$ ,  $\vec{e}_1 = (1, 0)^T, \vec{e}_2 = (0, 1)^T$  and  $m$  is an arbitrary natural number.

Note that the basis vectors of the invariant space  $\mathcal{I}$  are square-integrable on an interval  $(-\infty, B]$  with an arbitrary  $B < +\infty$ . It is also worth noting that there exists a QES scalar model of the same structure that has analogous properties [8].

By construction, QES operator (36) when restricted to the invariant space  $\mathcal{I}$  becomes complex  $2m \times 2m$  matrix  $M$ . However, the fact that operator (36) is Hermitian does not guarantee that the matrix  $M$  will be Hermitian. It is straightforward to check that the necessary and sufficient conditions of hermiticity of the matrix  $M$  read as

- basis vectors  $\vec{f}_j(y), \vec{g}_k(y)$  are square integrable on the interval  $[A, B]$ ,
- the condition

$$(\partial_y \vec{r}_j(y)) \vec{r}_k(y) - \vec{r}_j(y) (\partial_y \vec{r}_k(y))|_A^B = 0, \quad (37)$$

where  $\vec{r}_i = \vec{f}_i, i = 0, \dots, m-2$  and  $\vec{r}_i = \vec{g}_{i-m+1}, i = m-1, \dots, 2m-1$ , holds  $\forall j, k = 0, \dots, 2m-1$ .

In the case considered relation (37) does not hold and, consequently, the matrix  $M$  is not Hermitian. The next two examples are free of this drawback, since the basis vectors of their invariant spaces are square integrable on the interval  $(-\infty, +\infty)$ .

*Example 2.* Let us now consider Case I of the previous section with  $\alpha_1 = 1, \beta_2 = -1, \beta_0 = 1/2$ , the remaining coefficients being equal to zero. This choice yields the following QES matrix Schrödinger operator

$$\hat{H}[y] = \partial_y^2 - \frac{y^6}{256} + \frac{4m-1}{16}y^2 - \frac{1}{4}y^2\sigma_3 - \sigma_1.$$

The invariant space  $\mathcal{I}$  of this operator has the dimension  $2m$  and is spanned by the vectors

$$\begin{aligned}\vec{f}_j &= \exp\left(\frac{-y^4}{64}\right) \left(\frac{y}{2}\right)^{2j} \vec{e}_1, \\ \vec{g}_k &= \exp\left(\frac{-y^4}{64}\right) \left(m \left(\frac{y}{2}\right)^{2k} \vec{e}_2 - k \left(\frac{y}{2}\right)^{2k-2} \vec{e}_1\right),\end{aligned}$$

where  $j = 0, \dots, m-2, k = 0, \dots, m$ .

It is not difficult to verify that the basis vectors of the invariant space  $\mathcal{I}$  are square integrable on the interval  $(-\infty, +\infty)$  and that the corresponding matrix  $M$  is Hermitian. One more remark is that there exists an analogous QES scalar Schrödinger operator whose invariant space has square integrable basis vectors (see, for more details [2, 19]).

*Example 3.* Let us now consider Case III of the previous section with  $\alpha_2 = 1, \beta_2 = -1, \gamma_1 = -1$ . This choice of parameters yields the following QES matrix Schrödinger operator:

$$\begin{aligned}\hat{H}[y] &= \partial_y^2 - \frac{1}{4} - \frac{1}{4}\exp(-2y) + m\exp(-y) + \frac{1}{2}\exp(2y) \\ &+ \left[ m \frac{\sqrt{3}+1}{2} \sin(\sqrt{2}e^y) - \frac{\sqrt{6}}{2} \cos(\sqrt{2}e^y) - \exp(-y) \sin(\sqrt{2}e^y) \right] \sigma_1 \\ &+ \left[ m \frac{\sqrt{3}+1}{2} \cos(\sqrt{2}e^y) + \frac{\sqrt{6}}{2} \sin(\sqrt{2}e^y) - \exp(-y) \cos(\sqrt{2}e^y) \right] \sigma_3.\end{aligned}$$

Furthermore, the invariant space  $\mathcal{I}$  of this operator has the dimension  $2m$  and is spanned by the vectors

$$\begin{aligned}\vec{f}_j &= U^{-1}(y) \exp(-jy) \vec{e}_1, \\ \vec{g}_k &= U^{-1}(y) (m \exp(-ky) \vec{e}_2 - k \exp(-(k-1)y) \vec{e}_1),\end{aligned}$$

where  $j = 0, \dots, m-2, k = 0, \dots, m$ ,  $m$  is an arbitrary natural number and

$$\begin{aligned}U^{-1}(y) &= \frac{1}{2\sqrt{2}} \exp\left(-\frac{y}{2}\right) \exp\left(-\frac{1}{2}e^{-y}\right) \\ &\times (\sqrt{3} + \sqrt{2} - \sigma_3) \left[ \cos(\sqrt{2}e^y) + \frac{i\sqrt{3}\sigma_2 - \sigma_1}{\sqrt{2}} \sin(\sqrt{2}e^y) \right].\end{aligned}$$

The basis vectors of the invariant space  $\mathcal{I}$  are square integrable and the condition (37) holds. Indeed, the functions  $\vec{f}_j(y)$  and  $\vec{g}_k(y)$  behave asymptotically as  $\exp\left(-\frac{(2j+1)y}{2}\right)$  and



$\exp\left(-\frac{(2k+1)y}{2}\right)$ , correspondingly, with  $y \rightarrow +\infty$ . Furthermore, they behave as  $\exp\left(-\frac{(2j+1)y}{2}\right) \times \exp\left(-\frac{1}{2}e^{-y}\right)$  and  $\exp\left(-\frac{(2k+1)y}{2}\right) \exp\left(-\frac{1}{2}e^{-y}\right)$ , correspondingly, with  $y \rightarrow -\infty$ . This means that they vanish rapidly provided  $y \rightarrow \pm\infty$ .

*Example 4.* The last example to be presented here is the QES matrix model having a potential containing the Weierstrass function. To this end we consider the whole set of operators (16) and compose the Hamiltonian

$$H[x] = D_2 + A_1 + 2B_2.$$

Reducing  $H[x]$  to the Schrödinger form yields the following QES matrix model:

$$\begin{aligned} \hat{H}[y] = & \partial_y^2 + (m - m^2 - 1)w(y) - \frac{3(w(y)^2 - 1)^2}{16(w(y)^3 + w(y))} \\ & + \frac{2m - 1}{w(y)^2 + 1} (2\sigma_1 + (w(y)^3 + 3w(y))\sigma_3). \end{aligned}$$

Here  $m$  is an arbitrary natural number and  $w(y)$  is the Weierstrass function defined by the quadrature

$$y = \int_0^{w(y)} \frac{dx}{\sqrt{x^3 + x}}.$$

The invariant space  $\mathcal{I}$  of the operator  $\hat{H}[y]$  has the dimension  $2m$  and is spanned by the vectors

$$\begin{aligned} \vec{f}_j &= (w(y)^3 + w(y))^{-\frac{1}{4}} (1 - i\sigma_2 w(y)) \exp(-jy) \vec{e}_1, \\ \vec{g}_k &= (w(y)^3 + w(y))^{-\frac{1}{4}} (1 - i\sigma_2 w(y)) (m \exp(-ky) \vec{e}_2 \\ &\quad - k \exp(-(k-1)y) \vec{e}_1), \end{aligned}$$

where  $j = 0, \dots, m-2, k = 0, \dots, m$ .

Note that the first example of scalar QES model with elliptic potential has been constructed by Ushveridze (see [8] and the references therein).

## VI. Some conclusions

A principal aim of the paper is to give a systematic algebraic treatment of Hermitian QES Hamiltonians within the framework of the approach to constructing QES matrix models suggested in our papers [1, 9]. The whole procedure is based on a specific representation of the algebra  $o(2, 2)$  given by formulae (4), (6), (7). Making use of the fact that the algebra (7) has an infinite-dimensional invariant subspace (5) we have constructed in a systematic way six multi-parameter families of Hermitian QES Hamiltonians on line. Due to computational reasons we do not present here a systematic description of Hermitian QES Hamiltonians with potentials depending on elliptic functions (we give only an example of such Hamiltonian in Section V).

The problem of constructing all Hermitian QES Hamiltonians of the form (17) having square-integrable eigenfunctions is also beyond the scope of the present paper. We restricted our analysis of this problem to giving two examples of such Hamiltonians postponing its further investigation for our future publications.

A very interesting problem is a comparison of the results of the present paper based on structure of representation space of the representation (4), (6), (7) of the Lie algebra  $o(2, 2)$  to those of the paper [15], where some superalgebras of matrix-differential operators come into play. The link to the results of [15] is provided by the fact that the algebra  $o(2, 2)$  has a structure of a superalgebra. This is a consequence of the fact that operators (7) fulfill identities (8).

One more challenging problem is a utilization of the obtained results for integrating multi-dimensional Pauli equation with the help of the method of separation of variables. As an intermediate problem to be solved within the framework of the method in question is a reduction of the Pauli equation to four second-order systems of ordinary differential equations with the help of a separation Ansatz. The next step is studying whether the corresponding matrix-differential operators belong to one of the six classes of QES Hamiltonians constructed in Section IV.

Investigation of the above enumerated problems is in progress now and we hope to report the results obtained in one of our future publications.

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